## Last Time: Orthogonality.

### Gram-Schmidt Process:

Given lin. ind. vects. $v_1, v_2, \dots, v_k$ in $\mathbb{R}^n$, we can construct a set of mutually orthogonal vects $u_1, u_2, \dots, u_k$ with the same span. Formulaically:

$$\begin{cases} u_1 = v_1 \\ u_i := v_i - \text{proj}_{\boxed{u_{i-1}}}(v_i) - \text{proj}_{\boxed{u_{i-2}}}(v_i) - \cdots - \text{proj}_{\boxed{u_1}}(v_i) \end{cases}$$

**Ex:** Apply GS-process to $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

**Sol:** $u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$u_2 = v_2 - \text{proj}_{u_1}(v_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$u_3 = v_3 - \text{proj}_{u_2}(v_3) - \text{proj}_{u_1}(v_3)$$

$$= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + \frac{1}{2} - \frac{4}{3} \\ 3 + 0 - \frac{4}{3} \\ 1 - \frac{1}{2} - \frac{4}{3} \end{pmatrix} = \begin{pmatrix} -5/6 \\ 5/3 \\ -5/6 \end{pmatrix} = \frac{5}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

**Check:** $u_1 \cdot u_2 = 0$, $u_1 \cdot u_3 = 0$, $u_2 \cdot u_3 = 0$

$$u_1 \cdot u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0$$

$$u_1 \cdot u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{5}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \frac{5}{6}(-1 + 2 - 1) = \frac{5}{6} \cdot 0 = 0$$

$$u_2 \cdot u_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \frac{5}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \frac{5}{6}(-1 + 0 + 1) = \frac{5}{6} \cdot 0 = 0$$

Another check method: Note $u \cdot v = u^T v$

(i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = [xa + yb + zc]$ )

Take $A = [u_1 | u_2 | u_3]$, check

$$A^T A = \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline u_3^T \end{bmatrix} [u_1 | u_2 | u_3] = \begin{bmatrix} u_1^T u_1 & u_1^T u_2 & u_1^T u_3 \\ u_2^T u_1 & u_2^T u_2 & u_2^T u_3 \\ u_3^T u_1 & u_3^T u_2 & u_3^T u_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & u_1 \cdot u_3 \\ u_2 \cdot u_1 & u_2 \cdot u_2 & u_2 \cdot u_3 \\ u_3 \cdot u_1 & u_3 \cdot u_2 & u_3 \cdot u_3 \end{bmatrix} = \begin{bmatrix} |u_1|^2 & 0 & 0 \\ 0 & |u_2|^2 & 0 \\ 0 & 0 & |u_3|^2 \end{bmatrix}$$

if the $u_i$'s are mutually orthogonal...

Point: $A^T A$ is a diagonal matrix if columns of $A$ are mutually orthogonal... Should do: normalize the columns of $A$ (i.e. force $|u_i| = 1$ for all $i$ by taking suitable scalar multiples), then we obtain an "orthogonal matrix".

Def$^n$: A matrix $M$ is orthogonal when $M^T = M^{-1}$ ($M$ is $n \times n$).

Prop: $M$ is orthogonal if and only if the columns of $M$ form an orthonormal basis for $\mathbb{R}^n$. pf: Easy exercise $\boxed{/}$.

Def$^n$: A basis of $\mathbb{R}^n$ is orthonormal when the elements are mutually orthogonal and all have length 1.

Ex: Moment ago: we computed $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $u_3 = \frac{5}{6}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ form an orthogonal basis of $\mathbb{R}^3$. However,

$|u_1| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$, $|u_2| = \sqrt{1+1} = \sqrt{2}$,

$|u_3| = \frac{5}{6}\sqrt{1+4+1} = \frac{5}{6}\sqrt{6} = \frac{5}{\sqrt{6}}$, so $B = \{u_1, u_2, u_3\}$ is

not orthonormal... But

$$\hat{B} = \left\{ \frac{1}{|u_1|}u_1, \frac{1}{|u_2|}u_2, \frac{1}{|u_3|}u_3 \right\} = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{\frac{5}{6}}{5/\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix} \right\}$$ IS orthonormal :)

$$= \left\{ \begin{pmatrix}\frac{1}{\sqrt{3}}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix} \right\}$$

Check: $$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}^T \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \overset{???}{=} I_3$$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}+\frac{1}{3}+\frac{1}{3} & \frac{1}{\sqrt{2}\sqrt{3}}+0-\frac{1}{\sqrt{2}\sqrt{3}} & -\frac{1}{\sqrt{3}\sqrt{6}}+\frac{2}{\sqrt{3}\cdot\sqrt{6}}-\frac{1}{\sqrt{3}\sqrt{6}} \\ \frac{1}{\sqrt{2}\sqrt{3}}+0-\frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{2}+0+\frac{1}{2} & \frac{-1}{\sqrt{2}\sqrt{6}}+0+\frac{1}{\sqrt{2}\sqrt{6}} \\ -\frac{1}{\sqrt{3}\sqrt{6}}+\frac{2}{\sqrt{3}\sqrt{6}}-\frac{1}{\sqrt{3}\sqrt{6}} & \frac{-1}{\sqrt{2}\sqrt{6}}+0+\frac{1}{\sqrt{2}\sqrt{6}} & 1/6+\frac{4}{6}+\frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓ :)

$\boxed{//}$

Remark: This will always compute an orthonormal basis from orthogonal one.

Algorithm (Extended Gram-Schmidt Process): Given $v_1, v_2, \dots, v_k$ lin. indep. in $\mathbb{R}^n$. To compute an orthonormal collection w/ same span:

1. Apply the Gram-Schmidt Process to $v_1, v_2, \dots, v_k$.
2. Normalize each output vector (i.e. scale each $u_i$ by $\frac{1}{|u_i|}$).

Ex: Apply Extended GS process to $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(NB: Compare w/ previous example to note order matters for GS-process!)

Sol: $u_1 = v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$u_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \mathrm{proj}_{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + \frac{1}{2} \\ 3 + 0 \\ 1 - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3 \\ 1/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \mathrm{proj}_{\frac{1}{2}\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \mathrm{proj}_{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\cancel{\frac{1}{2}} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\cancel{\frac{1}{2}} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} \cdot \cancel{\frac{1}{2}} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}} \cancel{\frac{1}{2}} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{8}{38} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} - \frac{0}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{4}{19} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} = \frac{1}{19} \begin{pmatrix} 19 - 4 \\ 19 - 24 \\ 19 - 4 \end{pmatrix} = \frac{1}{19} \begin{pmatrix} 15 \\ -5 \\ 15 \end{pmatrix} = \frac{5}{19} \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

$\mathrm{Proj}_{c\vec{u}}(v) = \mathrm{Proj}_{u}(v)$

span is equal!

GS Process yields $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \boxed{\frac{1}{2}} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}, \boxed{\frac{5}{19}} \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \right\}$. Normalizing,

$$\hat{B} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{38}} \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{19}} \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \right\}.$$

$\boxtimes$

$|c\vec{u}| = |c||\vec{u}|$ so normalizing $c\vec{u} \rightsquigarrow \frac{1}{|c\vec{u}|} c\vec{u} = \frac{c}{|c||\vec{u}|}\vec{u} = \frac{1}{|\vec{u}|}\vec{u}$

$\hookrightarrow$ when $c > 0$

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Q: Why are we computing orthonormal bases?

A: Orthonormal collections give vectors very nice representations...

In the GS process: $u_i = v_i - \sum \mathrm{proj}_{u_j}(v_i)$

$$\rightsquigarrow v_i = \sum c_j u_j$$

when $u_j$'s orthonormal

$\ast$ So constant $c_j u_j = \mathrm{proj}_{u_j}(v_i) = \frac{u_j \cdot v_i}{u_j \cdot u_j} u_j = (u_j \cdot v_i) u_j$

**Ex:** The standard basis is an orthonormal basis.

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ae_1 + be_2 + ce_3 = (v\cdot e_1)e_1 + (v\cdot e_2)e_2 + (v\cdot e_3)e_3$$

**Point:** orthonormal bases generalize the standard basis.

**Ex:** Compute $\mathrm{Rep}_{\hat{B}}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ where $\hat{B} = \left\{ \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{u_1}, \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{u_2}, \underbrace{\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}}_{u_3} \right\}$.

**Sol:** $v = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ has

$$\left.\begin{aligned} u_1 \cdot v &= \tfrac{1}{\sqrt{3}}(2+1+2) = \tfrac{5}{\sqrt{3}} \\ u_2 \cdot v &= \tfrac{1}{\sqrt{2}}(2+0-2) = 0 \\ u_3 \cdot v &= \tfrac{1}{\sqrt{6}}(-2+2-2) = -\tfrac{2}{\sqrt{6}} \end{aligned}\right\} \rightsquigarrow \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{5}{\sqrt{3}}u_1 + 0u_2 - \frac{2}{\sqrt{6}}u_3 .$$

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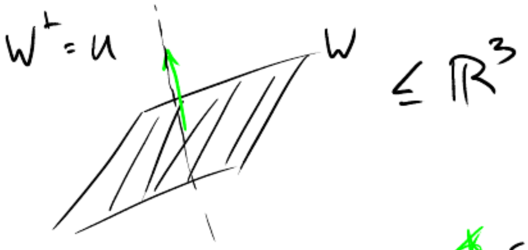
# ORTHOGONAL COMPLEMENTATION

**Defn:** A complement of subspace $W \leq V$ is a subspace $U$ such that every vector of $V$ can be expressed uniquely as $v = w + u$ where $w \in W$ and $u \in U$.

**Picture:**




**Prop:** If $W \leq \mathbb{R}^n$, then $W^\perp = \{u \in \mathbb{R}^n : u \cdot w = 0 \text{ for all } w \in W\}$ is the complement of $W$.

**Proof:** Every basis of $W$ extends to a basis of $\mathbb{R}^n$. Pick $B$ a basis of $W$. Apply Extended GS to obtain $\hat{B}$. $\hat{B}$ is still a basis of $W$. Extend to $A = \hat{B} \cup D$ a basis for $\mathbb{R}^n$. $\hat{A} = \hat{B} \cup \hat{D}$. $W = \mathrm{span}(\hat{B})$ and $W^\perp = \mathrm{span}(\hat{D})$. $\square$

Computationally: to compute $W^\perp$:

1. express $W = \mathrm{Col}(A)$ for matrix $A$.
2. $W^\perp = \mathrm{null}(A^T)$

Point → use $A$ = matrix of any basis